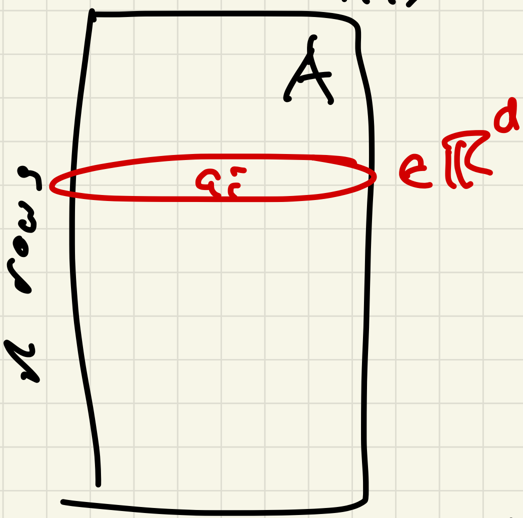
FoDA - L19

the Singular Value Decomposition (the SVD)

Input matrix $A \in \mathbb{R}^{n \times d}$

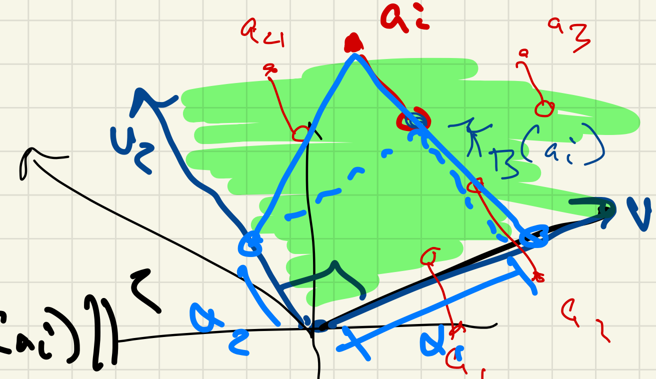
d columns



Goal B k-dimensional subspace

$V_B = \{v_1, v_2, \dots v_k\}$

- $\|v_i\| = 1$ $\quad v_i \in \mathbb{R}^d$
- $\langle v_i, v_j \rangle = 0 \quad j \neq i$



$$SSE(A,B) = \sum_{i=1}^{n} \|a_i - \pi_B(a_i)\|^2$$

$$\pi_B(a_i) = \sum_{j=1}^{k} \underbrace{\langle a_i, v_j \rangle}_{\alpha_j(a_i)} v_j$$

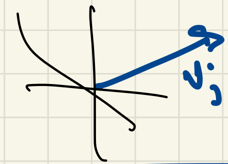
Singular Value Decomposition

$A = U S V^T$

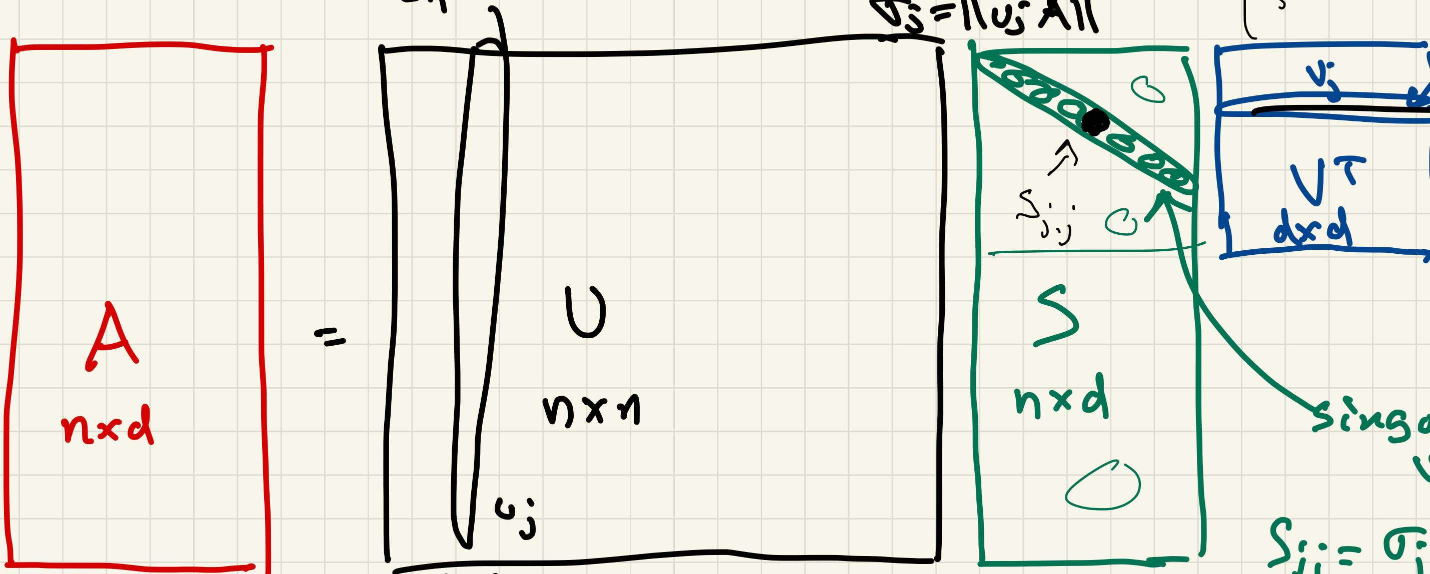
$A \in \mathbb{R}^{n \times d}$

$\sigma_j = \|A v_j\|$

$\sigma_j = \|u_j^T A\|$



$v_j \in \mathbb{R}^d$



right sing. vectors

$\|v_j\| = 1$

$\langle v_j, v_i \rangle = 0$

$j \neq i$

singular values

$S_{jj} = \sigma_j$

left sing. vectors.

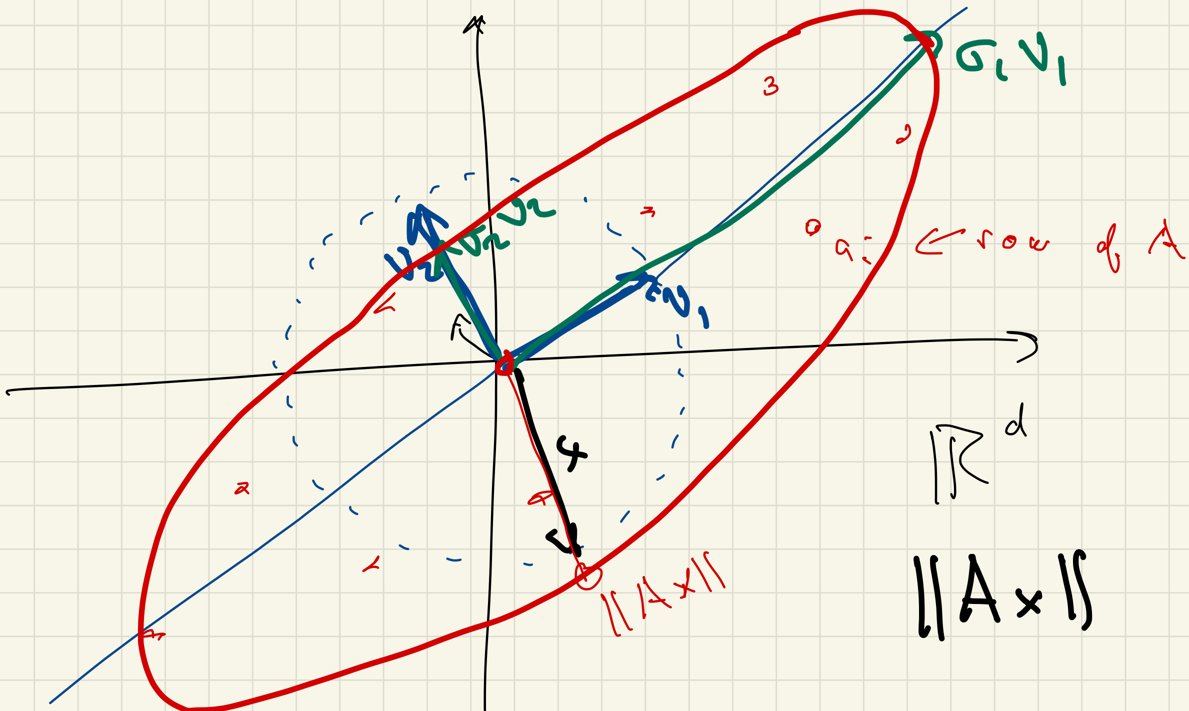
$\|u_j\| = 1$

$\langle u_j, u_i \rangle = 0 \quad i \neq j$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$

$r \leq d$

"shape" of A via SVD

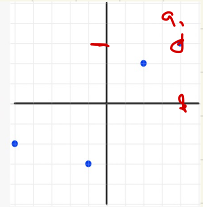


$\mathbb{R}^d$

$\|Ax\|$

Consider a matrix

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \\ -1 & -3 \\ -5 & -2 \end{pmatrix},$$



and its SVD $[U, S, V] = \mathrm{svd}(A)$:

$$U = \begin{pmatrix} -0.6122 & 0.0523 & 0.0642 & 0.7864 \\ -0.3415 & 0.2026 & 0.8489 & -0.3487 \\ 0.3130 & -0.8070 & 0.4264 & 0.2625 \\ 0.6408 & 0.5522 & 0.3057 & 0.4371 \end{pmatrix}, \quad S = \begin{pmatrix} 8.1655 & 0 \\ 0 & 2.3074 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -0.8142 & -0.5805 \\ -0.5805 & 0.8142 \end{pmatrix}.$$

$$\begin{array}{cc} v_1 & v_2 \end{array}$$

Unit vector

$x = (0.293, 0.970)$

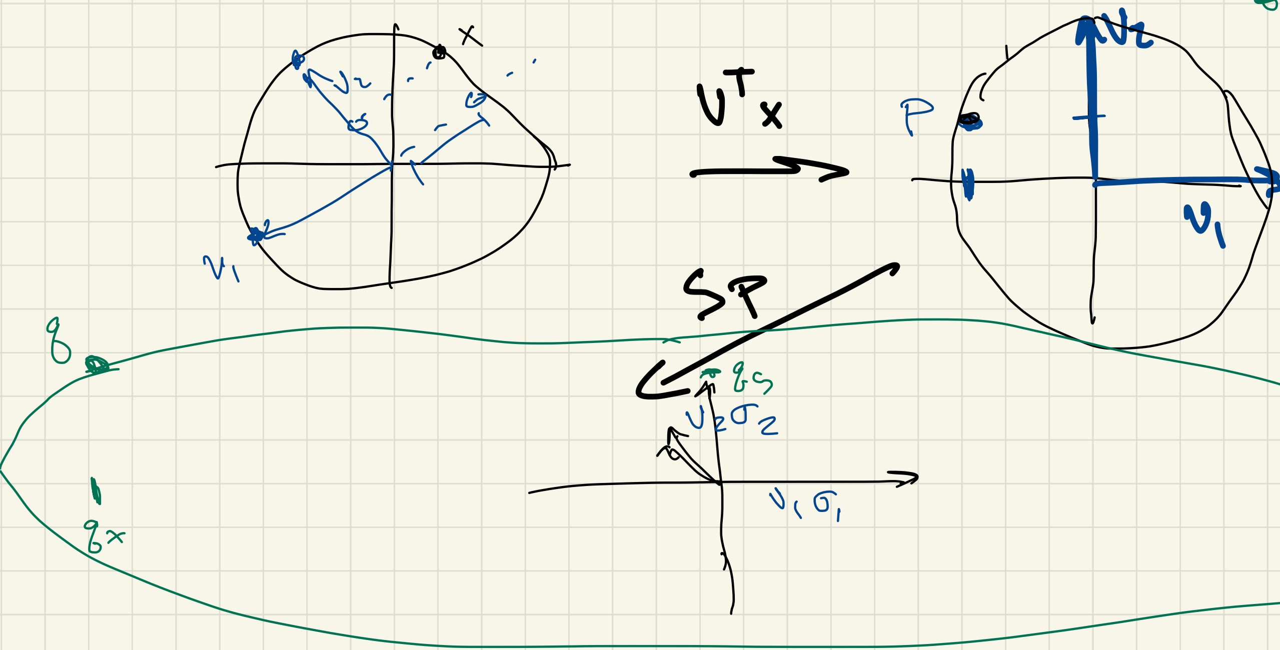
$Ax = USV^T x$

$p = V^T x$

q

$p = (\langle v_1, x\rangle, \langle v_2, x\rangle)$

$\|p\| = 1$



$q = SV^T x$

$q = Sp \in \mathbb{R}^4$

Best Rank-k Aprox

$V \quad B = \{v_1, v_2, \dots v_k\}$

$v_{k+1} \dots v_d$

↑

complete basis of $\mathbb{R}^d$

error for one data point $a \in \mathbb{R}^d$

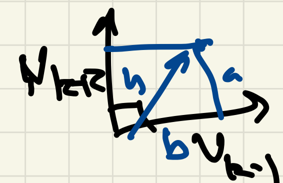
$$\|a - \pi_B(a)\|^2 = \Big\| \underbrace{\sum_{j=1}^{d} v_j \langle a, v_j \rangle}_{a} - \underbrace{\sum_{j=1}^{k} v_j \langle a, v_j \rangle}_{\pi_B(a)} \Big\|^2$$

$$= \Big\| \sum_{j=k+1}^{d} v_j \langle a, v_j \rangle \Big\|^2$$

Pythagorian

$$= \sum_{j=k+1}^{d} \langle a, v_j \rangle^2$$

$h^2 = a^2 + b^2$



$$\|A v\|^2 = \left\| \begin{bmatrix} \langle a_1, v \rangle \\ \langle a_2, v \rangle \\ \vdots \\ \langle a_n, v \rangle \end{bmatrix} \right\|^2 = \sum_{i=1}^{n} \langle a_i, v \rangle^2$$

$$\|A - \pi_B(A)\|^2 = \sum_{i=1}^{n} \|a_i - \pi_B(a_i)\|^2 = SSE(A,B)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=k+1}^{d} \langle a_i, v_j \rangle^2 \right) = \sum_{j=k+1}^{d} \|Av_j\|^2 = \sum_{j=k+1}^{d} \sigma_j^2$$

top k RSVecs of A $= \underset{\forall\, B = \{v_1, \ldots v_k\}\ k\text{-dim}}{\operatorname{argmin}}\ SSE(A,B)$

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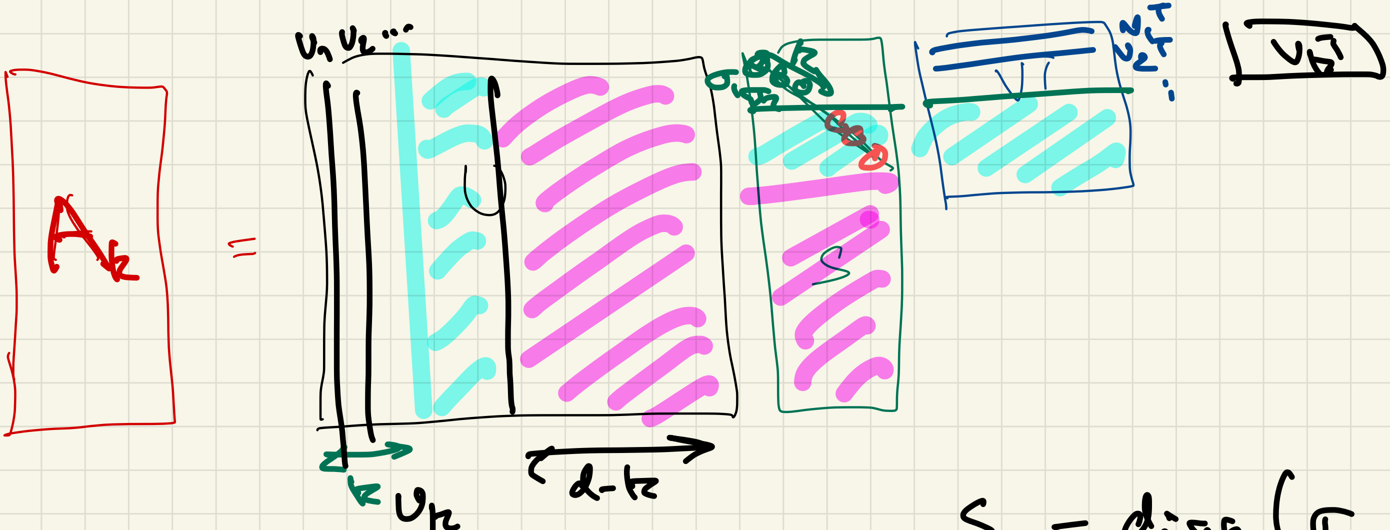
rank of $B = k$

best $\min \|A - A_k\|_F$ or $\min \|A - A_k\|_2$

$A_k =$ best rank k approximation of A

$$A_k = \sum_{j=1}^{k} \underbrace{\sigma_j}_{\text{sing value}} \overbrace{u_j}^{\text{LSV}} \underbrace{v_j^T}_{\text{RSV}} = \underbrace{\sigma_1 u_1 v_1^T}_{\in \mathbb{R}^{n \times d},\ \text{rank } 1} + \sigma_2 u_2 v_2^T + \cdots$$



$$S_k = \mathrm{diag}(\sigma_1 \dots \sigma_k)$$

$n \times d$

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$$

or

$$= U S_k V^T$$

$$= U_k S_k V_k^T$$

$$\|A - A_k\|_F^2 = \sum_{j=k+1}^{d} \sigma_j^2$$

$$\|A - A_k\|_2^2 = \max_{\|x\|=1} \|(A - A_k)x\|^2$$

$$= \sigma_{k+1}^2$$