FoDA L8: Linear Algebra Review #1: Vectors, Matrices, Addition, Multiplication

Vectors & Matrices

<u>vector</u> $v = (v_1, v_2, \dots, v_d) \in \mathbb{R}^d$

Scalar

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} \qquad v^T = [v_1 \;\; v_2 \;\; \dots \;\; v_d]$$

column vector — row vector

n data points

d dimension / attributes

<u>matrix</u> $n \times d$ matrix $A \in \mathbb{R}^{n \times d}$

n rows $a_1, a_2, \dots a_n \in \mathbb{R}^d$

$$A = [a_1 ; a_2 ; \dots ; a_n] = \begin{bmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix} = \begin{bmatrix} A_{11}, A_{12}, \dots A_{1d} \\ A_{21}, A_{22} \dots A_{2d} \\ \vdots \\ A_{n1} \; A_{n2} \dots A_{nd} \end{bmatrix}$$

vector

$$v = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 5 \end{bmatrix} \in \mathbb{R}^4$$

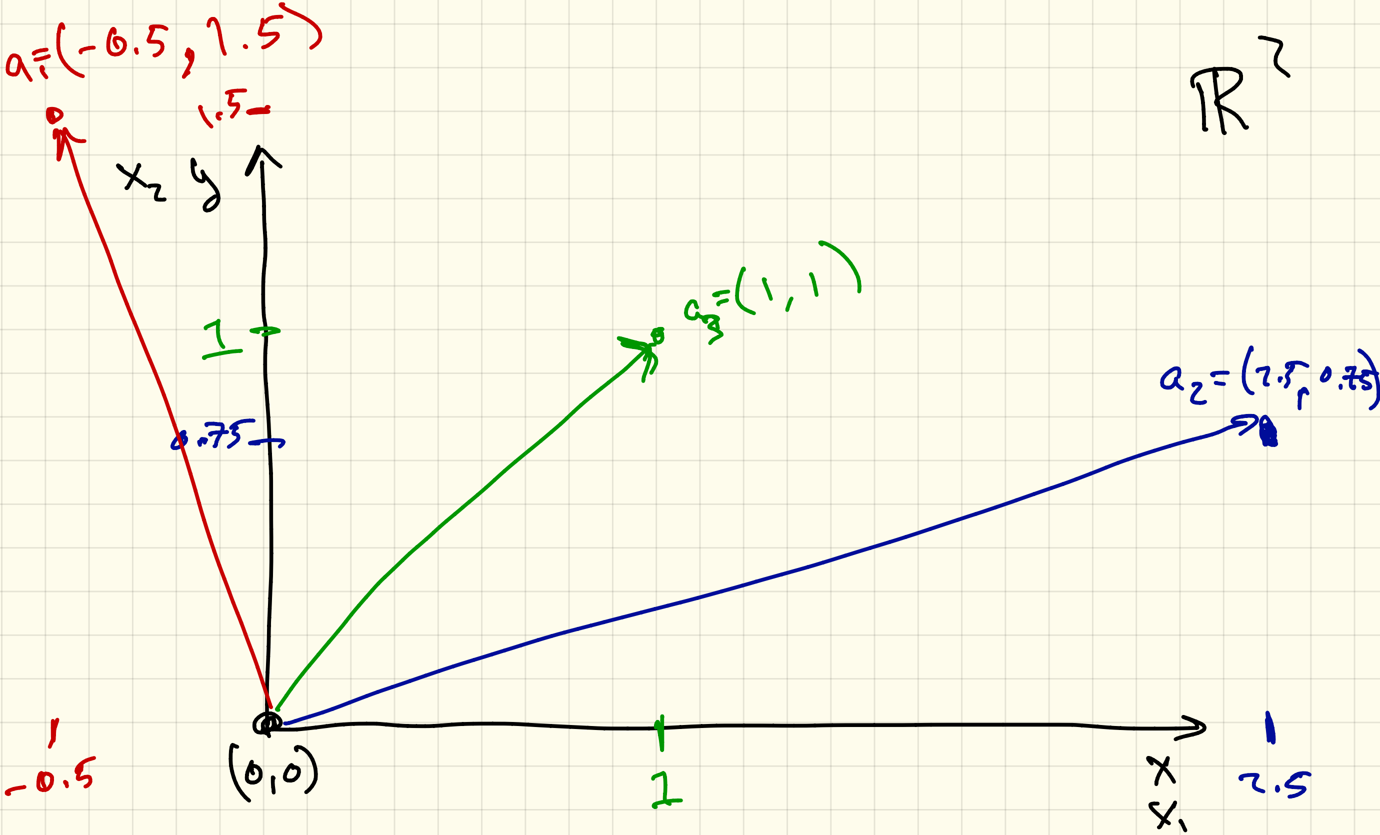
matrix

$$a_1 = (3, -7, 2) \in \mathbb{R}^3$$

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

transpose : change roll of rows & columns

$$A^T = \begin{bmatrix} 3 & -1 \\ -7 & 2 \\ 2 & -5 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$$



$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.5 & 1.5 \\ 2.5 & 0.75 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$$

Linear Equations

$n = 2$ eq
$d = 3$ variables

$$3x_1 - 7x_2 + 2x_3 = -2$$
$$-1x_1 + 2x_2 - 5x_3 = 6$$

$$Ax = b$$

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

$$b = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \in \mathbb{R}^{2 \times 1} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

Addition

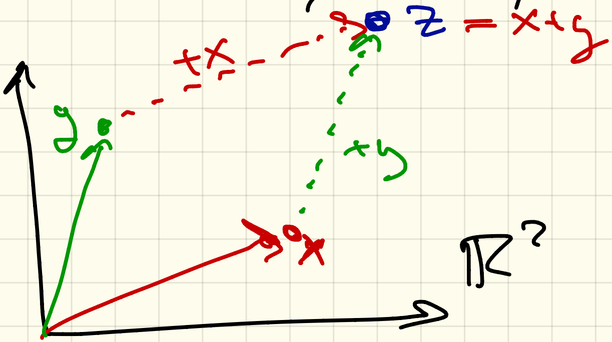
element-wise

$x = (x_1, x_2, \dots x_d)$ $y = (y_1, y_2, \dots y_d)$

$x, y \in \mathbb{R}^d$ \$\mathbb(R)^d\$

\in

$z = x + y = (x_1 + y_1, x_2 + y_2, \dots, x_d + y_d) \in \mathbb{R}^d$



$\mathbb{R}^2$

$$A, B \in \mathbb{R}^{n \times d}$$

$$C = A + B \implies C_{ij} = A_{ij} + B_{ij}$$

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 8 & 1 \end{bmatrix}$$

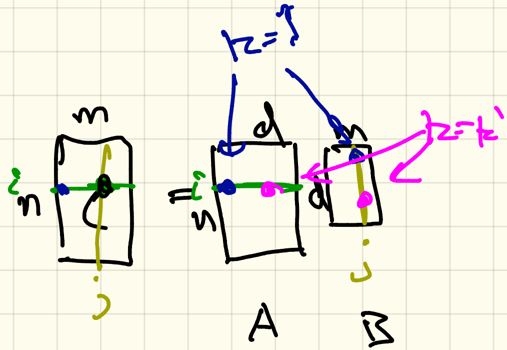
$$C = A + B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 10 & -4 \end{bmatrix}$$

Multiplication

$A \in \mathbb{R}^{n \times d}$ $\quad$ $B \in \mathbb{R}^{d \times m}$

$C = AB \in \mathbb{R}^{n \times m}$

$$C_{ij} = \sum_{k=1}^{d} A_{ik} B_{kj}$$



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require # col in A = # rows in B

A, B, C matrices $\quad$ AB legal $\quad$ maybe BA not legal (unless n=m)

associative $\quad (AB)C = A(BC)$

distributive $\quad A(B+C) = AB + AC$

not commutative $\quad AB \neq BA$

$$G = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} \in \mathbb{R}^{2\times 2} \qquad B = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 8 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

$$GB = \begin{bmatrix} 14 & 29 & 6 \\ 36 & 76 & 19 \end{bmatrix}$$

$$(GB)_{11} = 1\cdot 2 + 3\cdot 4$$
$$2 + 12 = 14$$

$$B = \alpha A \qquad A, B \in \mathbb{R}^{n\times d}$$

$\alpha \in \mathbb{R} \leftarrow$ scalar

$B_{ij} = \alpha \cdot A_{ij}$ elementwise

Vector-vector multiplications

inner product and outer product

$x \in \mathbb{R}^n \quad y \in \mathbb{R}^m$ column vectors

outer product

OK if $n \neq m$

$$x y^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [y_1 \ \dots \ y_m] = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \dots & x_1 y_m \\ \\ x_n y_1 & & & x_n y_m \end{bmatrix}$$

$$\in \mathbb{R}^{n \times m}$$

Inner (dot) product

$x, y \in \mathbb{R}^d$ column vectors

\langle   \rangle

$$x^T y = x \cdot y = \langle x, y \rangle = [x_1 \ldots x_d] \begin{bmatrix} y_1 \\ \vdots \\ y_d \end{bmatrix} = \sum_{i=1}^{d} x_i y_i \in \mathbb{R}$$

output is scalar

associative, distributive, commutative

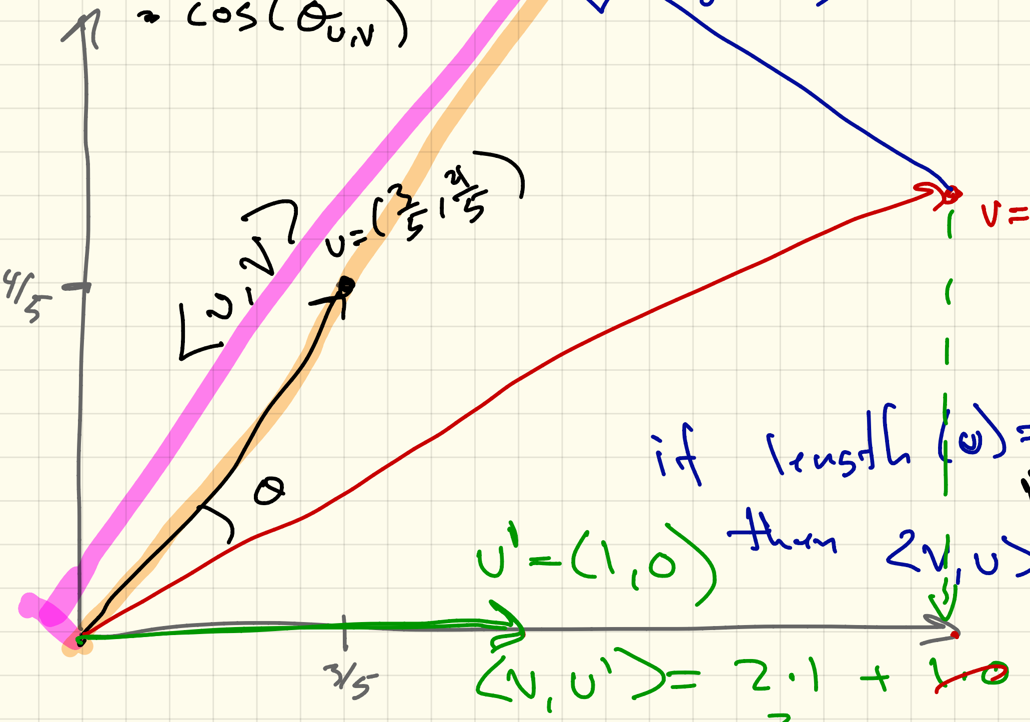
$x, y, z \in \mathbb{R}^d \qquad \alpha \in \mathbb{R}$

$$\langle \alpha x, y + z \rangle = \alpha \langle x, y + z \rangle = \alpha \left(\langle x, y \rangle + \langle x, z \rangle \right)$$

Dot Product

$$\langle u, v\rangle = \text{length}(u)\cdot \text{length}(v)\cdot \cos(\theta_{u,v})$$

1 $\quad u = \left(\frac{3}{5}, \frac{4}{5}\right)$ $\quad v = (2, 1)$ $\quad \pi_u(v)$



if $\text{length}(u) = \|u\| = 1$ $\quad \mathbb{R}^2$

then $\langle v, u\rangle = \|\pi_u(v)\|$

$u' = (1, 0)$

$\langle v, u'\rangle = 2\cdot 1 + \cancel{1\cdot 0}$
$= 2$

Matrix - vector Multiplication

$A \in \mathbb{R}^{n \times d}$ $\quad$ $x \in \mathbb{R}^d$

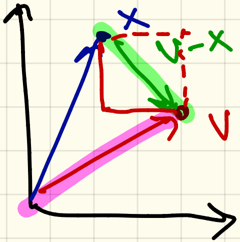
$$y = Ax \in \mathbb{R}^n$$

$$A = \begin{bmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix} \qquad a_i \in \mathbb{R}^d$$

$$y = Ax = \begin{bmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix} x = \begin{bmatrix} \langle a_1, x \rangle \\ \langle a_2, x \rangle \\ \vdots \\ \langle a_n, x \rangle \end{bmatrix} \in \mathbb{R}^n$$

Norms (Vectors)

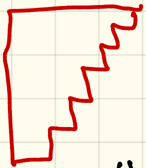
"how big?"



vector $v = (v_1, v_2, \dots, v_d) \in \mathbb{R}^d$

$$\|v\| = \|v\|_2 = \sqrt{\sum_{i=1}^{d} v_i^2} = \sqrt{v_1 v_1 + v_2 v_2 + \dots + v_d v_d} = \sqrt{\langle v, v \rangle}$$

$$\|v - x\| = \sqrt{\sum_{i=1}^{d} (v_i - x_i)^2}$$ = Euclidean distance



$$\|v\|_p = \left(\sum_{i=1}^{d} |v_i|^p \right)^{1/p}$$ any $p \in [1, \infty]$

$p=1$

$$\|v\|_1 = \sum_{i=1}^{d} |v_i|$$

$$\|v\|_\infty = \max_{i \in [1,d]} |v_i|$$