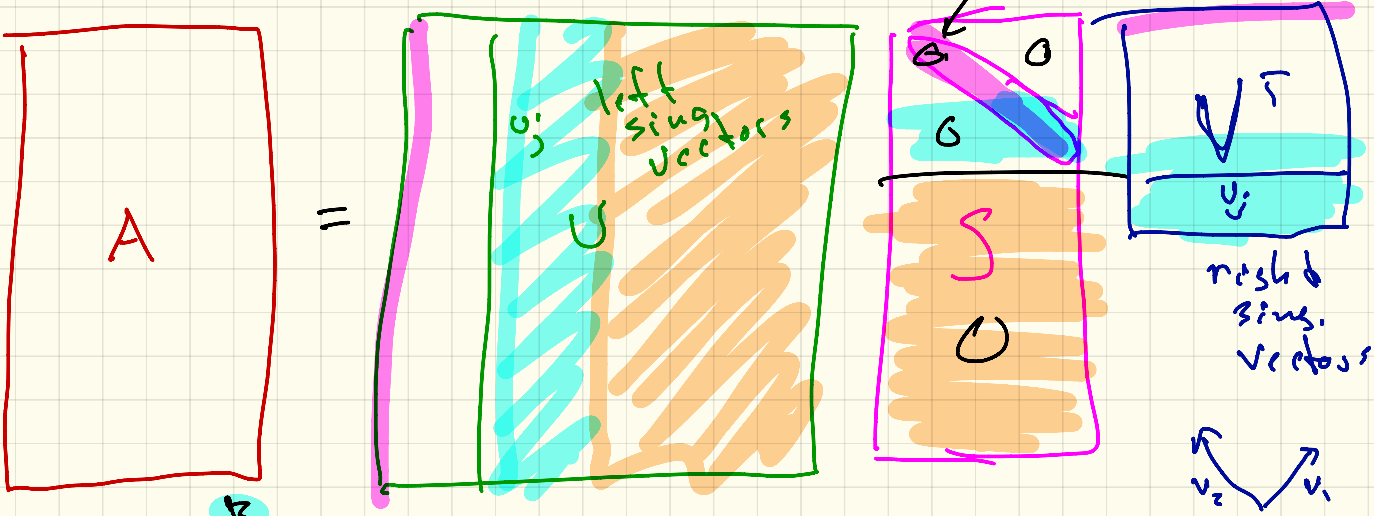
FoDA L20 : the Power Method

Input $A \in \mathbb{R}^{n \times d}$ $\quad A \subset \mathbb{R}^d$

$\{a_1, a_2, \dots a_n\}$

Data Matrix $\quad \|a_i - a_j\|$ makes sense

σ_j: sing values



$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$$

$\text{rank}(A_k) = k$

$\sigma_1 \geq \sigma_2 \geq \dots \sigma_r$

top k

most important

Power Method

Input $A \in \mathbb{R}^{n \times d}$ Output top right sing. vector v_1

$M = A^T A \in \mathbb{R}^{d \times d}$ positive semi-definite

top right svec (A) = top eigenvector (M)

$u \leftarrow$ random unit vector in $\mathbb{R}^d$

$V = M^8 u$

for $i = 1$ to 8
$u^{(i)} = M u^{(i-1)}$

return $v_1 = \frac{v}{\|v\|_2}$

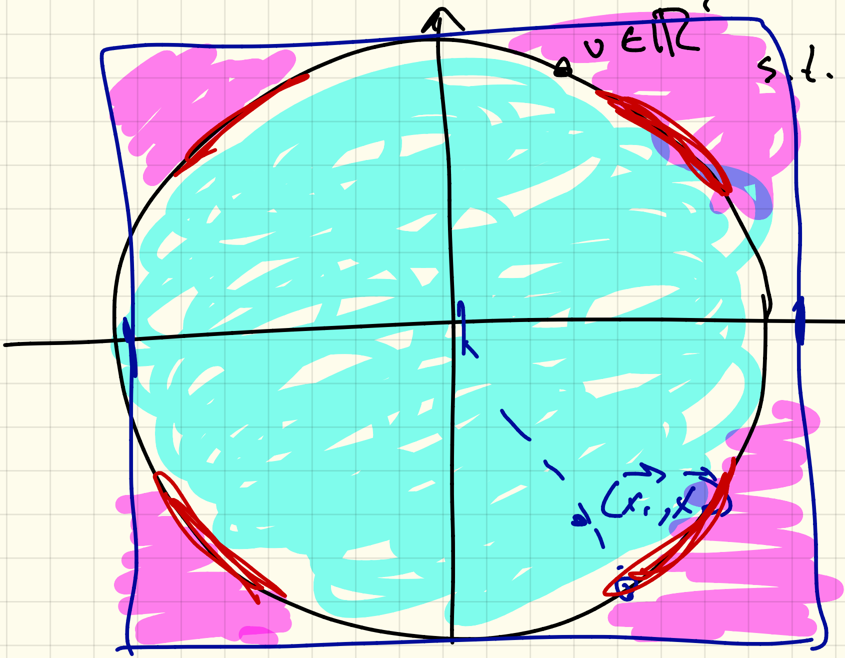
$v = M(M(\dots(Mu)).)$

matrix-vector multiply $\underbrace{}_{}$ 8 times

$M^8 = M \cdot M \cdot \dots \cdot M$

matrix-matrix mult

Random Unit Vector



$v \in \mathbb{R}^2$ s.t. $\|v\| = 1$

$g \sim \text{Gaussian}_d(0, 1)$

$\uparrow$ Box-Muller (Unif)

$$v = \frac{g}{\|g\|}$$

$x \in \text{Unif}(0,1)$

$x_1 \sim \text{Unif}(0,1) \rightarrow (2x_1 - 1) = \vec{x}_1$

$x_2 \sim \text{Unif}(0,1) \rightarrow (2x_2 - 1) = \vec{x}_2$

Power Method ($M = A^TA$, q)

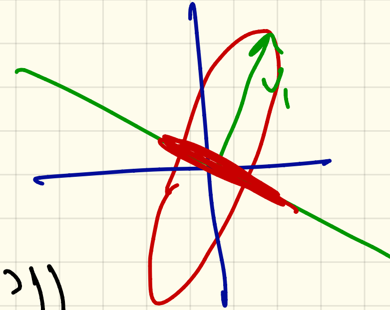
0. Init $u^{(0)} \leftarrow$ random Gaussian

1. for $i = 1$ to q

   $u^{(i)} = M u^{(i-1)}$

2. Return $v = u^{(q)} / \|u^{(q)}\|$

top eigenvector v_1 of M



$$\sigma_1^2 = \lambda_1 = \|M v_1\|$$

$M v_1 = \lambda_1 v_1$

$\|v_1\| = 1$

$\|M v_1\| = \|\lambda_1 v_1\| = \lambda_2$

$A_1 = A - A v_1 v_1^T$

$M_1 = A_1^T A_1$

$v_2 =$ power-method(M_1) $\Rightarrow$ recurse

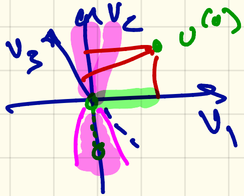
$M = A^T A$, assume full rank $\quad M \in \mathbb{R}^{d \times d}$

eigenvectors $M = \{v_1, v_2, \dots v_d\}$

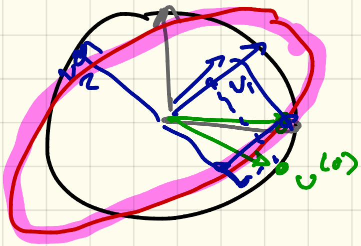
eigenvalues $L = \{\lambda_1, \lambda_2, \dots \lambda_d\}$

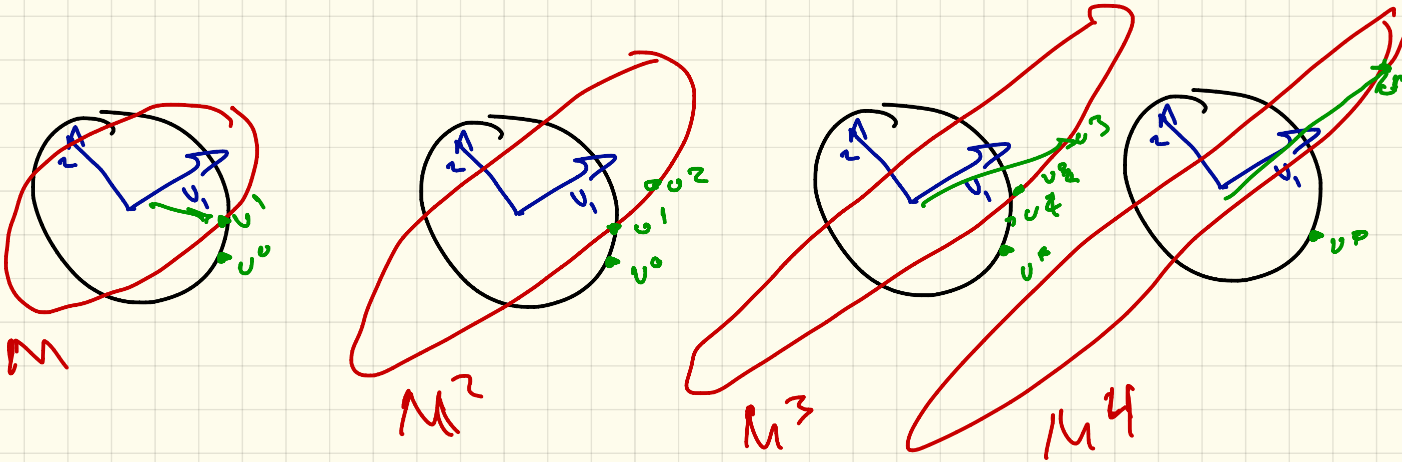
Init vector $\quad u^{(0)} \in \mathbb{R}^d$

$$u^{(0)} = \sum_{j=1}^{d} \alpha_j v_j \qquad \alpha_j = \langle u^{(0)}, v_j \rangle$$



assume $\quad |\alpha_1| \geq \frac{1}{2} \frac{1}{\sqrt{d}}$




M
M²
M³
M⁴
v₁
v₂
u⁰
u¹
u²
u³

eigen values of $M^8 = \lambda_1^8, \lambda_2^8, \lambda_3^8 \dots \lambda_d^8$

vector $M^8 = v_1, v_2, \dots v_d$

$$M^8 v_j = M \cdot M \cdot \dots (M v_j) = M^{(8-1)} (v_j \lambda_j)$$

$$= (M^{8-2})(v_j \lambda_j) \lambda_j$$

$$= M^{8-3} (v_j \lambda_j) \lambda_j^2$$

$$\dots$$

$$= M v_j \lambda_j^{8-1}$$

$$= v_j \lambda_j^8$$

$$\alpha_j = \langle v_j , u^{(0)} \rangle$$

$$\hat{v}_1 = \frac{M^q u^{(0)}}{\|M^q u^{(0)}\|} = \frac{\sum_{j=1}^{d} \alpha_j \lambda_j^q v_j}{\sqrt{\sum_{j=1}^{d} (\alpha_j \lambda_j^q)^2}}$$

$$\langle v_j , v_i \rangle = 0$$

$$i \neq j$$

$$|\langle \hat{v}_1 , v_1 \rangle| = \frac{\alpha_1 \lambda_1^q}{\sqrt{\sum_{j=1}^{d} (\alpha_j \lambda_j^q)^2}} \geq \frac{\alpha_1 \lambda_1^q}{\sqrt{\alpha_1^2 \lambda_1^{2q} + d\, \alpha_2^2 \lambda_2^{2q} + \dots}}$$

$$\alpha_1 > \frac{1}{2} \frac{1}{\sqrt{d}}$$

$$\geq \frac{\alpha_1 \lambda_1^q}{\alpha_1 \lambda_1^q + \lambda_2^q \sqrt{d}} = 1 - \frac{\lambda_2^q \sqrt{d}}{\alpha_1 \lambda_1^q + \cancel{\lambda_2^q \sqrt{d}}}$$

$$\geq 1 - \frac{\lambda_2^q \sqrt{d}}{\alpha_1 \lambda_1^q} \geq 1 - 2d \left(\frac{\lambda_2}{\lambda_1} \right)^q$$

Convergence of Power Method

goes exponentially fast in

$$\frac{\lambda_2}{\lambda_1}$$

$$\text{error} \approx \left(\frac{\lambda_2}{\lambda_1}\right)^{q}$$